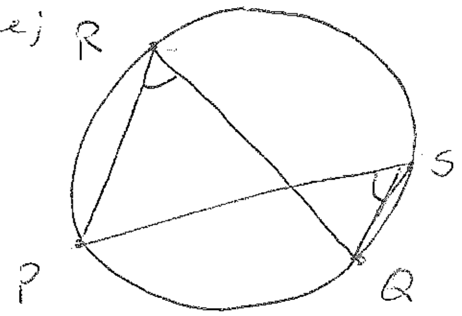
1.

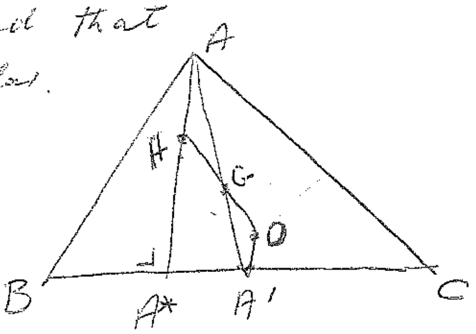
Centres of Triangles   9 March   UCD

Here are the notes which we didn't get to, on 9th March.
We will continue the same notation:
[AA'], [BB'] and [CC'] denote the medians of $\triangle ABC$,
[AA*], [BB*], [CC*] " " altitudes " ",
O is the circumcircle of " ";
G is the centroid of " ",
H is the orthocentre of " ".

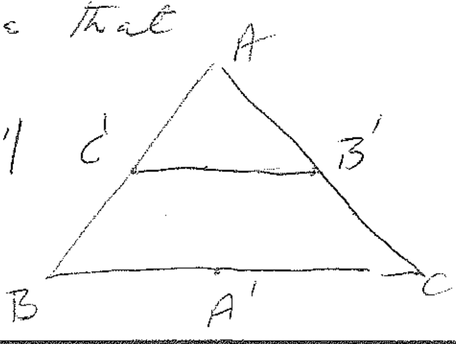
Recall:
(A) The theorem in J.C. that states that if PQ is an arc of a circle and R and S are points on the circle, then $|\angle PRQ| = |\angle PSQ|$.
The converse is also true;
if $|\angle PRQ| = |\angle PSQ|$
then PRSQ is a cyclic quadrilateral.



Recall:
(B) When we proved that H, G and O are collinear (on the Euler line) we showed that the $\triangle$s AHG and OGA' are similar.
We noted that $|AH| = 2|OA'|$, which was irrelevant then but we will need now.



Recall:
(C) The J.C. Theorem which states that the $\triangle$s AC'B' and ABC are similar, so $2|C'B'| = |BC| = 2|BA'|$ and $C'B' \parallel BC$.



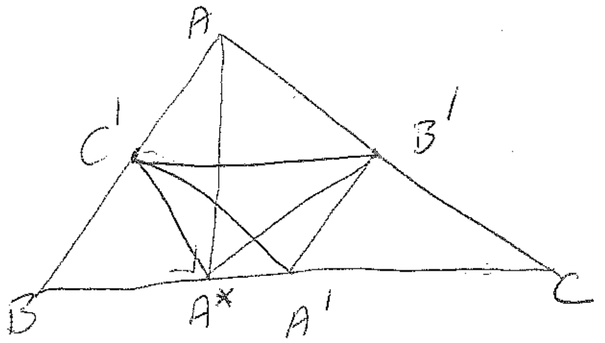
2. We will show that

(a) A^*, B^* and C^* lies on the circumcircle of $\triangle A'B'C'$ making a "six-point circle", sometimes called Feuerbach's circle,

(b) three other points, which we will define on the altitudes, lie on the same circle — "the nine-point circle" it is usually called,

c) the centre, N of the nine-point circle lies on the Euler line.

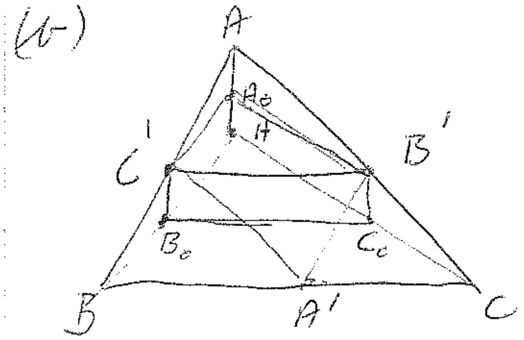
(a) Our first step is to show that $A'B'C'A^*$ is a cyclic quadrilateral.



As an exercise, show that
$\triangle AC'B'$ is congruent to $\triangle B'C'A'$ ($AB'A'C'$ is a cyclic quadrilateral) and
$\triangle AC'B'$ is congruent to $\triangle A^*C'B'$ ($C'B'$ is the perpendicular bisector of $[AA^*]$, so reflect in $C'B'$).
So, $\triangle B'C'A'$ and $\triangle A^*C'B'$ are congruent, with $|\angle C'A^*B'| = |\angle C'A'B'|$. We conclude from Theorem (A) on page 1 that $A'B'C'A^*$ is a cyclic quadrilateral.
That is, A^* lies on the circumcircle of $\triangle A'B'C'$.

3. Similarly, we can show that B^* and C^* also lie on the circumcircle of $\triangle A'B'C'$.
Now, we have six-points of the nine-point circle.

(b)

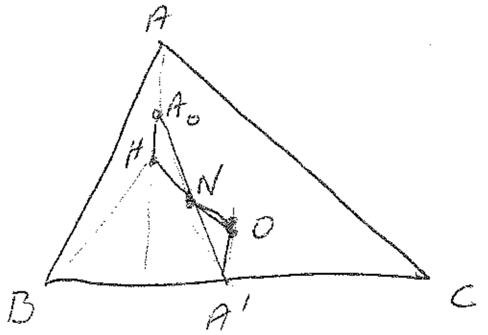


A_0 is the midpoint of $[AH]$,
B_0 " . . . . . " $[BH]$,
C_0 " " " " $[CH]$.

We will show that A_0, B_0 and C_0 lie on the same circle, the nine-point circle.
Consider the $\triangle BAH$. Since $|AC'| = |C'B|$ and $|HB_0| = |B_0B|$, we can conclude from Theorem 3 on page 1 that $C'B^0 \parallel AH$.
Similarly, by considering the $\triangle CAH$, we have $B'C^0 \parallel AH$. Thus, $C'B^0 \parallel B'C^0$.
From $\triangle BHC$, we have $B^0C^0 \parallel BC$.
Note also that $AH \perp BC$ (AH is an altitude).
Therefore, $C'B'C_0B_0$ is a rectangle and so is cyclic with diameter $[CC_0]$.
Similarly, we can show that $C'A_0 \parallel A_1C_0$ and $CA_1 \parallel A_0C$, so $C'A_0B'A'$ is also a rectangle with diameter $[CC_0]$ for its circumcircle, i.e. the same diameter. So, those two rectangles have a common circumcircle. In other words, A_0, B_0 and C_0 lie on the circumcircle of $\triangle A_1B_1C_1$ which is the nine-point circle.
We'll call the centre of this circle N.

4. Another "$\triangle$ centre" which was discovered (in 1986) to lie on the Euler line is the Exeter point, You have a diagram showing its construction from the 9th March class in U.C.D. There are many more.

(c) We will prove that the "triangle centre" N that is the centre of the nine point circle, also lies on the Euler line.



From page 3, we know that $[C_0C']$, $[A_0A']$ and $[B_0B']$ are diameters of the nine-point circle. So N is the midpoint of, say, $[A_0A']$.

Consider the $\triangle A_0HN$ and the $\triangle A'ON$.

Now, $|A_0H| = |OA'|$, (see page 1, Recall B)

$|A_0N| = |AN|$ and $|\angle HA_0N| = |\angle NA'O|$ $(AH \parallel OA')$

So these $\triangle$s A_0HN and $A'ON$ are congruent.

Thus ~~the~~ $|\angle HNA_0| = |\angle ONA'|$. We know that A_0, N and A' are collinear, so these are truly opposite angles. Therefore H, N and O must be collinear. That is, N lies on the Euler line.

Recall that $3|OG| = |OH|$.

We also have from the above that $|HN| = |NO|$, so, $2|ON| = |OH| = 3|OG|$. The line segments connecting the points of the Euler line that we've met, have their lengths in small natural number ratios to each other.

I hope you all are continuing to enjoy maths.

Best wishes,

Mary Hanly